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A NOTE ON POLYNOMIAL MATRIX FUNCTIONS OVER A FINITE FIELD.(U)

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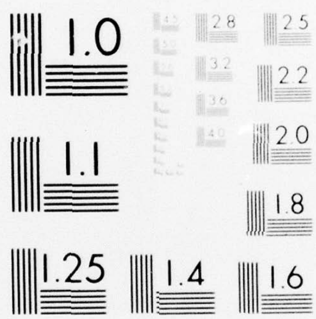
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CLEMSON UNIVERSITY
Clemson, South Carolina

A NOTE ON POLYNOMIAL MATRIX FUNCTIONS OVER A FINITE FIELD

BY

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A Note on Polynomial Matrix Functions

over a Finite Field

by J.V. Brawley*

1. Let $F = GF(q)$ denote the finite field of order q, and let F_n denote the ring of nxn matrices over F. Consider an element $A(x) \varepsilon F_n[x]$; i.e.,

$$A(x) = A_N x^N + A_{N-1} x^{N-1} + \ldots A_1 x + A_0 \tag{1}$$

where $A_i \varepsilon F_n$. This polynomial defines via substitution several functions from F_n to F_n. Two such functions are

$$B \rightarrow A_r(B) = A_N B^N + A_{N-1} B^{N-1} + \ldots + A_1 B + A_0 \tag{2}$$

and

$$B \rightarrow A_L(B) = B^N A_N + B^{N-1} A_{N-1} + \ldots + BA_1 + A_0 \text{ .} \tag{3}$$

We call (2) and (3), respectively, the right and left polynomial functions determined by $A(x)$ with the terms right and left indicating the side on which the substituting variable is placed.

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Definition. A function $A: F_n \to F_n$ is called a right respectively left) polynomial function if there exists a polynomial $A(x) \in F_n[x]$ which represents A via the right subsitution (2) (respectively(3)).

In this note we obtain unique representations for and determine the number of right (left) polynomial functions $A: F_n \to F_n$. Proofs will be given for the right functions which can be obviously modified for the left polynomial functions.

2. Recall that

$$L_n(x) = \prod_{i=1}^{n} (x^{q^i} - x) \tag{4}$$

is the monic polynomial of least degree in F[x] satisfied by every $B \in F_n$; indeed, $L_n(x)$ is the least common multiple of all degree n polynomials in F[x] [See, 2]. We define δ by

$$\delta = \deg L_n(x) = q^n + q^{n-1} + \ldots + q. \tag{5}$$

THEOREM 1. Let $Z(x) = \sum_{i=0}^{N} Z_i x^i$ be a polynomial in $F_n[x]$ with $\deg Z(x) = N < \delta$. If $Z_r(B) = Z_N B^N + \ldots + Z_1 B + Z_0 = 0$ for every $B \in F_n$, then $Z_i = 0$, $i = 0, 1, 2, \ldots, N$.

Proof. Let $f(x) = x^n - a_{n-1}x^{n-1} - \ldots - a_1 x - a_0$ be an arbitrary polynomial of degree n in F[x], and let $C \in F_n$ denote the companion matrix of f(x). Dividing Z(x) by f(x) we obtain

$$Z(x) = Q(x)\, f(x) + R(x) \tag{6}$$

where $Q(x)$ and $R(x)$ are in $F_n[x]$ with

$$R(x) = R_{n-1}x^{n-1} + \ldots + R_1x + R_0 \ . \tag{7}$$

Since $f(x)$ is a scalar polynomial we may substitute an arbitrary matrix B into (6) to get $Z_r(B) = Q_r(B)f(B) + R_r(B)$. In particular, for every nonsingular $P \in GL(n,q)$ it follows from the Hamilton-Cayley theorem that

$$0 = Z_r(PCP^{-1}) = R_r(PCP^{-1}) \ .$$

Thus $(R_r(PCP^{-1}))P = 0$ or

$$R_{n-1}PC^{n-1} + R_{n-2}PC^{n-1} + \ldots + R_1PC + R_0P = 0 \tag{8}$$

for every $P \in GL(n,q)$.

Now it is known [1] that each matrix $X \in F_n$ can be written as a linear combination of nonsingular matrices P_i; i.e.,

$$X = c_1P_1 + c_2P_2 + \ldots + c_tP_t \ , \ c_i \in F.$$

If follows from (8) that

$$R_{n-1}XC^{n-1} + R_{n-2}XC^{n-1} + \ldots + R_1XC + R_0X = 0 \tag{9}$$

for every $X \in F_n$. In particular, if we take $X = E_m$ where E_m has a 1 in position $(m,1)$ and zeros elsewhere we find through actual computation that equation (9) reduces to

$$\begin{pmatrix} r_{1m}^{(0)} & r_{1m}^{(1)} & \cdots & r_{1m}^{(n-1)} \\ r_{2m}^{(0)} & r_{2m}^{(1)} & \cdots & r_{2m}^{(n-1)} \\ \vdots & \vdots & & \vdots \\ r_{nm}^{(0)} & r_{nm}^{(1)} & & r_{nm}^{(n-1)} \end{pmatrix} = 0$$

where $R_k = (r_{ij}^{(k)})$. Thus column m of R_k is zero for $k = 0, 1, \ldots, n-1$ and $m = 1, 2, \ldots, n$; i.e., $R_k = 0$ for $k = 0, 1, \ldots, n-1$. It follows from (6) that $f(x)$ divides $Z(x)$ for every monic of degree n; hence $L_n(x)$ divides $Z(x)$. But $\deg Z(x) < \deg L_n(x)$ so $Z(x)$ must be the zero polynomial; i.e., every $Z_i = 0$ and the proof is complete.

As a corollary to Theo[illegible] we have the following:

THEOREM 2. <u>Each right polynomial function</u> $A:F_n \to F_n$ <u>can be represented uniquely by a polynomial</u> $A(x) \in F_n[x]$ <u>of degree</u> $< \delta$ <u>and each such polynomial represents a right polynomial function. The number of right polynomial functions is therefore</u> $q^{n^2\delta}$.

Proof. If $A_1(x)$ and $A_2(x)$ have degree $< \delta$ and each represent the right polynomial function A then $A_1(x) - A_2(x)$ represents the zero function; hence by Theorem 1, $A_1(x) = A_2(x)$.

Finally let A be a right polynomial function and let $A(x)$

represent A. By division

$$A(x) = Q(x)L_n(x) + R(x)$$

where $R(x)$ has degree $< \delta$. Clearly, $R(x)$ represents A.

References

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2. J. V. Brawley, L. Carlitz, and Jack Levine. Scalar polynomial functions on the $n \times n$ matrices over a finite field. Linear Algebra and Its Applications. 10(1975), 199-217.

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13. ABSTRACT

Let $F = GF(q)$ denote the finite field of order q, and let F_n denote the ring of $n \times n$ matrices over F. Each matrix polynomial $A(x) = A_N x^N + \ldots + A_1 x + A_0$ in $F_n[x]$ defines via substitution several functions from F_n to F_n. Two such functions, called respectively, the right and left polynomial functions determined by A(x) are

$$B \to A_r(B) = A_N B^n + \ldots + A_1 B + A_0$$

$$B \to A_L(B) = B^n A_N + \ldots + BA_1 + A_0$$

A function $A: F_n \to F_n$ is called a right (left) polynomial function if there exists $A(x) \in F_n[x]$ which represents A via the right (left) substitution $B \to A_r(B)$ $(B \to A_L(B))$. This paper obtains a unique representation for and determines the number of right (left) polynomial functions $A: F_n \to F_n$.

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